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S_3 symmetry and neutrino masses and mixings

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Abstract. Based on the universal seesaw mass matrix model with the three scalars ϕ_i, and by assuming S_3 flavor symmetry for the Yukawa interactions, the lepton masses and mixings are investigated systematically. In order to understand the observed neutrino mixing, the charged leptons (e, μ, τ) are regarded as the three objects (e_1, e_2, e_3) of S_3, while the neutrino mass eigenstates are regarded as the irreducible representation $(\nu_\eta, \nu_\sigma, \nu_\pi)$ of S_3, where (ν_π, ν_η) and ν_σ are a doublet and a singlet, respectively, which are composed of the three objects (ν_1, ν_2, ν_3) of S_3.

1 Introduction

Generally masses and mixings of the quarks and leptons are considered to obey a simple law of nature, so that we expect that we will find a beautiful relation among those values. However, even if there is such a simple relation in the quark sector, it is hard to see such a relation in the quark sector, because the relation will be spoiled by the gluon cloud. We may expect that such a beautiful relation will be found only in the lepton sector. Therefore, in the present paper, we will confine ourselves to the investigation of the lepton masses and mixings. Here, we would like to emphasize that we should search a model that gives a reasonable description of not only the masses, but also of the mixings. Especially, we should direct our attention to the mixing pattern rather than to the mass spectrum in the neutrino sector.

Also, the mass matrices of the fundamental particles are considered to be governed by a symmetry. In the present paper, we take the permutation symmetry S_3 [1–4]. Let us begin by giving a short review of how useful a description may be that is based on the S_3 symmetry for the lepton masses and mixings.

The observed neutrino data have strongly suggested that the neutrino mixing is approximately described by the so-called tribimaximal mixing [5–14]:

$$U_{\rm TB} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{1}$$

The tribimaximal mixing is interpreted in the framework of S_3: we define the doublet (ψ_π, ψ_η) and the singlet ψ_σ of the permutation symmetry S_3 as

$$\begin{pmatrix} \psi_\pi \\ \psi_\eta \\ \psi_\sigma \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} , \tag{2}$$

where (ψ_1, ψ_2, ψ_3) are the three objects of S_3. When we assume that the mass eigenstates in the charged lepton sector are $(e_1, e_2, e_3) = (\tau, \mu, e)$ while those in the neutrino sector are $(\nu_\pi, \nu_\eta, \nu_\sigma)$, with the mass hierarchy

$$m^2_{\nu\eta} < m^2_{\nu\sigma} < m^2_{\nu\pi} , \tag{3}$$

the neutrino mixing matrix U_ν of the basis $(\nu_\eta, \nu_\sigma, \nu_\pi)$ to the basis $(e_1, e_2, e_3) = (e, \mu, \tau)$ is given by the form (1), because the basis $(\nu_\eta, \nu_\sigma, \nu_\pi)$ is given by

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \equiv \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \nu_\eta \\ \nu_\sigma \\ \nu_\pi \end{pmatrix} . \tag{4}$$

Here, the weak iso-doublets are given by $(\nu_i, e_i)_{\rm L}$ $(i = 1, 2, 3)$ [and also $(\nu_a, e_a)_{\rm L}$ $(a = \pi, \eta, \sigma)$]. In other words, in order to obtain the tribimaximal mixing, we must build a model in which the mass eigenstates are $(e_1, e_2, e_3) = (\tau, \mu, e)$ and $(\nu_\pi, \nu_\eta, \nu_\sigma)$ with the mass hierarchy (3).

On the other hand, it is well known that the observed charged lepton mass spectrum [15] satisfies the relation [16–19]

$$m_e + m_\mu + m_\tau = \frac{2}{3}\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2 \tag{5}$$

with remarkable precision. The mass formula (5) is invariant under any exchange $\sqrt{m_i} \leftrightarrow \sqrt{m_j}$ $(i, j = e, \mu, \tau)$. This,

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too, suggests that a description by S_3 may be useful for a mass matrix model.

As an explanation of the mass formula (5), the author has proposed a model [19–21] with the three flavor scalars ϕ_i in the framework of the universal seesaw model[1] [23–38]: the fermion mass matrix M_f is given by

$$M_f = m_{\rm L}^f M_F^{-1} m_{\rm R}^f , \tag{6}$$

where M_F is the mass matrix of the hypothetical heavy fermions F_i $(i = 1, 2, 3)$. For example, for the charged lepton sector, we assume

$$m_{\rm L}^e = \frac{1}{\kappa} m_{\rm R}^e = y_e {\rm diag}(v_1, v_2, v_3) \tag{7}$$

(κ is a constant with $\kappa \gg 1$), and $M_E \propto \mathbf{1} \equiv {\rm diag}(1, 1, 1)$, where $v_i \equiv \langle \phi^0_{{\rm L}i} \rangle = \langle \phi^0_{{\rm R}i} \rangle / \kappa$, and $m_{\rm L}^e$ (and also $m_{\rm R}^e$) is defined by $\bar{e}_{\rm L} m_{\rm L}^2 E_{\rm R}$.

If we assume that the vacuum expectation values (VEVs) v_i satisfy the relation

$$v_1^2 + v_2^2 + v_3^2 = \frac{2}{3}(v_1 + v_2 + v_3)^2, \tag{8}$$

we can obtain the relation (5). Of course, here we have assumed that the Yukawa interaction in the charged lepton sector is given by an S_3 invariant form:

$$H_e = y_e \left(\bar{\ell}_{\rm L1} \phi_{\rm L1} E_{\rm R1} + \bar{\ell}_{\rm L2} \phi_{\rm L2} E_{\rm R2} + \bar{\ell}_{\rm L3} \phi_{\rm L3} E_{\rm R3} \right) \tag{9}$$

(and also a similar interaction for $\bar{\ell}_{\rm R} \phi_{\rm R} E_{\rm L}$), where $\ell_{\rm L/R} = (\nu_{\rm L/R}, e_{\rm L/R})$, and $\phi_{\rm L/R} = (\phi^+_{\rm L/R}, \phi^0_{\rm L/R})$. Equation (9) is not of the general form under S_3 symmetry. We have assumed universality of the coupling constants in addition to S_3 symmetry.

The relation among the VEVs v_i, (8), may read

$$v_\pi^2 + v_\eta^2 = v_\sigma^2 \tag{10}$$

in terms of S_3, because

$$v_1^2 + v_2^2 + v_3^2 = v_\pi^2 + v_\eta^2 + v_\sigma^2 = 2 v_\sigma^2 = 2 \left(\frac{v_1 + v_2 + v_3}{\sqrt{3}} \right)^2 , \tag{11}$$

where $v_a = \langle \phi_a^0 \rangle$ $(a = \pi, \eta, \sigma)$, and $(\phi_\pi, \phi_\eta, \phi_\sigma)$ have been defined by (4). For a Higgs potential model based on the S_3 symmetry that leads to the relation (10), see, for example, see [39, 40]. The S_3 symmetry is again related to the lepton masses and mixings.

Thus, it is likely that the S_3 symmetry (or a higher symmetry, which includes S_3) plays an essential role in the unified description of the lepton mass matrices. In the present paper, we will assume that, in the universal seesaw model with three scalars ϕ_i, the Yukawa interactions are exactly invariant under the S_3 symmetry, and the S_3 symmetry is broken only by the VEVs v_i of the three scalars ϕ_i. For the seesaw mass matrix model (6), by inheriting the formulation in the charged lepton sector, we assume the following. (i) The M_F have a unit matrix structure, at least for the charged lepton and neutrino sectors, i.e.

$$M_E \propto \mathbf{1} , \quad M_N \propto \mathbf{1} . \tag{12}$$

(ii) The $m_{\rm L}^f$ (and $m_{\rm R}^f$) have a sector-dependent (f-dependent) structure. We still assume the diagonal form

$$m_{\rm L}^e = y_e {\rm diag}\left(v_1^d, v_2^d, v_3^d\right) \tag{13}$$

in the charged lepton sector, but we consider $m_{\rm L}^\nu$ in the neutrino sector not to be diagonal. Therefore, in the present model (6), the neutrino mixing is caused by the structure of $m_{\rm L}^\nu$. We also assume that the VEVs v_i^u satisfy the relation (8) as well as v_i^d in the charged lepton sector. However, note that, in spite of the assumption (8) for the v_i^u, the eigenvalues of the matrix $m_{\rm L}^\nu$, in general, do not satisfy a relation similar to (8). The purpose of the present paper is to investigate what structure of the Dirac mass matrix $m_{\rm L}^\nu$ in the seesaw mass matrix model (6) is required in order to fit the model for the neutrino oscillation data.

By the way, the seesaw-type model (6) with three scalars $\phi_{{\rm L}i}$ (and $\phi_{{\rm R}i}$) causes some trouble; for example, we have the flavor changing neutral current (FCNC) problem, spoiling of the asymptotic freedom of the SU(3) color, and so on. Therefore, instead of the Yukawa interaction (9), we may consider a Frogatt–Nielsen-type model [41] with the five dimensional operators $\bar{\ell}_{{\rm L}i} H_{\rm L} \phi_i E_{{\rm R}i}$, where $H_{\rm L}$ is the conventional Higgs scalar ${\rm SU(2)_L}$-doublet $H_{\rm L} = (H_{\rm L}^+, H_{\rm L}^0)$, and the ϕ_i are the three-family ${\rm SU(2)_L}$-singlet scalars:

$$H_{\rm eff} = y_e \bar{\ell}_{\rm L} H_{\rm L}^d \frac{\phi^d}{\Lambda_d} E_{\rm R} + y_\nu \bar{\ell}_{\rm L} H_{\rm L}^u \frac{\phi^u}{\Lambda_u} N_{\rm R} , \tag{14}$$

where the Λ_f are the scales of the effective theory. We consider m_W^2 to obey $m_W^2 = \frac{1}{2} g_w^2 (\langle H_d^0 \rangle^2 + \langle H_u^0 \rangle^2)$ and further $\langle \phi^f \rangle / \Lambda_f \sim 1$ $(f = u, d)$. Since we are interested only in the flavor structure, for convenience, hereafter, we will drop the Higgs scalars $H_{\rm L}^f$ from (14), and we will simply call $\bar{\ell}_{\rm L} H_{\rm L}^u \phi^u N_{\rm R}$ the Yukawa interaction $\bar{\ell}_{\rm L} \phi^u N_{\rm R}$.

2 Mass eigenvalues

In general, the S_3 invariant Yukawa interaction with the three scalars ϕ_a $(a = \pi, \eta, \sigma)$ is given by

$$\begin{aligned} H = & \left(y_0 \frac{\bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta + \bar{\psi}_\sigma \psi_\sigma}{\sqrt{3}} \right. \\ & \left. + y_1 \frac{\bar{\psi}_\pi \psi_\pi + \bar{\psi}_\eta \psi_\eta - 2 \bar{\psi}_\sigma \psi_\sigma}{\sqrt{6}} \right) \phi_\sigma \\ & + y_2 \left(\frac{\bar{\psi}_\pi \psi_\eta + \bar{\psi}_\eta \psi_\pi}{\sqrt{2}} \phi_\pi + \frac{\bar{\psi}_\pi \psi_\pi - \bar{\psi}_\eta \psi_\eta}{\sqrt{2}} \phi_\eta \right) \\ & + y_3 \frac{\bar{\psi}_\pi \phi_\pi + \bar{\psi}_\eta \phi_\eta}{\sqrt{2}} \psi_\sigma + y_4 \bar{\psi}_\sigma \frac{\phi_\pi \psi_\pi + \phi_\eta \psi_\eta}{\sqrt{2}} , \end{aligned} \tag{15}$$

where we have $\bar{\psi} = \bar{\ell}_{\rm L} \equiv (\bar{\nu}_{\rm L}, \bar{e}_{\rm L})$, $\psi = E_{\rm R}$ and $\phi_a = \phi_a^d$ for

[1] The seesaw mechanism for charged particles is known as the "universal seesaw mechanism" [22].

the charged lepton sector, $\bar{\psi}=\bar{\ell}_{\rm L}$, $\psi=N_{\rm R}$ (or $\nu_{\rm R}$) and $\phi_a=\phi_a^u$ for the neutrino sector, and we have dropped $H_{\rm L}^f/\Lambda_f$ for convenience. For example, the interaction (9) in the charged lepton sector corresponds to the case

$$y_0=y_e\,,\qquad y_1=0\,,\qquad y_2=\frac{1}{\sqrt{3}}y_e\,,\qquad y_3=y_4=\sqrt{\frac{2}{3}}y_e\,. \tag{16}$$

The Yukawa interaction (15) gives the mass matrix $m_{\rm L}^f$ for the basis $(\psi_\pi,\psi_\eta,\psi_\sigma)$,

$$m_{\rm L}^f=\begin{pmatrix} \left(\frac{y_0}{\sqrt{3}}+\frac{y_1}{\sqrt{6}}\right)v_\sigma+\frac{y_2}{\sqrt{2}}v_\eta & \frac{y_2}{\sqrt{2}}v_\pi & \frac{y_3}{\sqrt{2}}v_\pi \\ \frac{y_2}{\sqrt{2}}v_\pi & \left(\frac{y_0}{\sqrt{3}}+\frac{y_2}{\sqrt{6}}\right)v_\sigma-\frac{y_2}{\sqrt{2}}v_\eta & \frac{y_3}{\sqrt{2}}v_\eta \\ \frac{y_4}{\sqrt{2}}v_\pi & \frac{y_4}{\sqrt{2}}v_\eta & \left(\frac{y_0}{\sqrt{3}}-2\frac{y_1}{\sqrt{6}}\right)v_\sigma \end{pmatrix}. \tag{17}$$

Hereafter, for simplicity, we confine ourselves to the investigation of the case with a symmetric mass matrix form $(m_{\rm L}^f)^{\rm T}=m_{\rm L}^f$, i.e. with $y_3=y_4$. Then we still have five parameters, $y_0v_\sigma, y_1v_\sigma, y_2v_\pi, y_3v_\pi$ and v_π/v_η, in the model, so that the model has no predictability. In the present paper, we do not impose further symmetry on the model. Alternatively, we will investigate what constraints on the mass matrix parameters (or specific relations among those) are required from the phenomenological studies.

Now let us return to the subject of the neutrino Dirac mass matrix $m_{\rm L}^\nu$, which is, in general, given by (17) on the basis $(\nu_\pi,\nu_\eta,\nu_\sigma)$. (Hereafter, for convenience, we will simply denote y_i^ν as y_i.) As we discussed in the previous section, the present neutrino oscillation data favor tribimaximal mixing, so that the neutrino states are approximately in the mass eigenstates $(\nu_\eta,\nu_\sigma,\nu_\pi)$ with $m_\eta^2<m_\sigma^2<m_\pi^2$. Therefore, for convenience, we investigate the case of the limit of $y_3=0$. (Since the observed neutrino mixing is not the exact tribimaximal mixing, the condition $y_3=0$ is only an approximate requirement, posed for convenience.) The mass matrix with $y_3=0$ is diagonalized by a rotation

$$R(\theta_{\pi\eta})=\begin{pmatrix} c_{\pi\eta} & s_{\pi\eta} & 0 \\ -s_{\pi\eta} & c_{\pi\eta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

where $c_{\pi\eta}=\cos\theta_{\pi\eta}$ and $s_{\pi\eta}=\sin\theta_{\pi\eta}$, and

$$\tan 2\theta_{\pi\eta}=-\frac{v_\pi}{v_\eta}\,, \tag{19}$$

as

$$R^{\rm T}(\theta_{\pi\eta})m_{\rm L}^\nu R(\theta_{\pi\eta})={\rm diag}(m_\pi,m_\eta,m_\sigma)\,. \tag{20}$$

The mass eigenvalues m_π, m_η and m_σ are given by

$$m_\pi=\left(\frac{y_0}{\sqrt{3}}+\frac{y_1}{\sqrt{6}}\right)v_\sigma\pm\frac{|y_2|}{\sqrt{2}}\sqrt{v_\pi^2+v_\eta^2}\,,$$
$$m_\eta=\left(\frac{y_0}{\sqrt{3}}+\frac{y_1}{\sqrt{6}}\right)v_\sigma\mp\frac{|y_2|}{\sqrt{2}}\sqrt{v_\pi^2+v_\eta^2}\,,$$
$$m_\sigma=\left(\frac{y_0}{\sqrt{3}}-2\frac{y_0}{\sqrt{6}}\right)v_\sigma\,, \tag{21}$$

where we have defined

$$\sqrt{2}y_0+y_1>0\,, \tag{22}$$

and the upper and lower signs in $\pm|y_2|$ (and also in $\mp|y_2|$) correspond to the cases $y_2v_\eta>0$ and $y_2v_\eta<0$, respectively.

In the previous section, we have assumed that the VEVs v_i^d of the scalars ϕ_i^d, which couple to the charged leptons, satisfy the relation (10). Therefore, we also assume that the VEVs v_i^u of the scalar ϕ_i^u, which couple to the neutrino sector, satisfy the relation

$$\left(v_\pi^u\right)^2+\left(v_\eta^u\right)^2=\left(v_\sigma^u\right)^2\equiv\frac{1}{2}v_u^2\,, \tag{23}$$

where we do not always consider $\langle\phi_i^u\rangle=\langle\phi_i^d\rangle$. Then the mass eigenvalues (21) lead to

$$m_\pi=\left(\frac{1}{\sqrt{6}}y_0+\frac{1}{2\sqrt{3}}y_1\pm\frac{1}{2}|y_2|\right)v_u\,,$$
$$m_\eta=\left(\frac{1}{\sqrt{6}}y_0+\frac{1}{2\sqrt{3}}y_1\mp\frac{1}{2}|y_2|\right)v_u\,,$$
$$m_\sigma=\left(\frac{1}{\sqrt{6}}y_0-\frac{1}{\sqrt{3}}y_1\right)v_u\,. \tag{24}$$

Note that the mass spectrum is independent of the parameters v_π^u/v_σ^u and v_η^u/v_σ^u and only depends on the parameters y_1/y_0 and $|y_2|/y_0$. On the other hand, as seen in (19), the mixing angle $\theta_{\pi\eta}$ is independent of the parameters y_i and only depends on the parameter v_π^u/v_η^u.

As we discussed in Sect. 1, the observed tribimaximal mixing suggests that the neutrino mass eigenstates are $(\nu_\eta,\nu_\sigma,\nu_\pi)$. If the mass hierarchy is of normal type, it demands that $m_\eta^2<m_\sigma^2\ll m_\pi^2$, and if it is of inverse type, it demands that $m_\pi^2\ll m_\eta^2<m_\sigma^2$. The conditions for $m_\eta^2<m_\sigma^2<m_\pi^2$ and $m_\pi^2<m_\eta^2<m_\sigma^2$ are given in the appendix.

By the way, we still have two adjustable parameters y_1/y_0 and y_2/y_0 to predict the neutrino mass spectrum. In the following sections, we will investigate two typical cases by making assumptions for the coupling constants y_0, y_1 and y_2. Of course, the assumptions must also be applicable to the charged lepton coupling constants (16).

3 Case with $y_0^2=y_1^2+y_2^2$

In the mass matrix (17), the y_1- and y_2-terms are traceless, while the trace of the y_0-term is not zero. This suggests that the y_0-term may be distinguished from the other terms under a higher symmetry. Therefore, by way of trial, we pose the following normalization condition for the coupling constants:

$$y_0^2=y_1^2+y_2^2+y_3^2\,, \tag{25}$$

which is satisfied by the coupling constants (16) in the charged lepton sector. Since we have assumed that $y_3=0$ in the neutrino sector, we can explicitly write the condition (25) as

$$y_1 = y_0 \sin\alpha\,, \quad y_2 = y_0 \cos\alpha\,. \tag{26}$$

Then we can rewrite (24) as follows:

$$\begin{aligned} m_\pi &= \left[\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\sin\left(\alpha \mp \frac{2}{3}\pi\right)\right] y_0 v_u\,, \\ m_\eta &= \left[\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\sin\left(\alpha \pm \frac{2}{3}\pi\right)\right] y_0 v_u\,, \\ m_\sigma &= \left[\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\sin\alpha\right] y_0 v_u\,, \end{aligned} \tag{27}$$

where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ ($\cos\alpha > 0$), and, for $\frac{\pi}{2} \le \alpha < \frac{3}{2}\pi$, we substitute $\pi - \alpha$ for α in (27).

Note that the case with the condition (25) that leads to (27) gives the relation

$$m_\pi^2 + m_\eta^2 + m_\sigma^2 = \frac{2}{3}(m_\pi + m_\eta + m_\sigma)^2. \tag{28}$$

Since these masses $(m_\pi, m_\eta, m_\sigma)$ are Dirac masses in the neutrino seesaw mass matrix $M_\nu = m_L^\nu M_N^{-1}(m_L^\nu)^T$, if we take the heavy Majorana mass matrix M_N with the unit matrix form, we obtain neutrino masses that are proportional to m_π^2, m_η^2 and m_σ^2, respectively. Therefore, the neutrino masses will satisfy a relation similar to the charged lepton mass relation (1).

The differences among m_π^2, m_η^2 and m_σ^2 are given as follows:

$$m_\pi^2 - m_\eta^2 = \pm\frac{1}{\sqrt{3}}\cos\alpha\left(\sqrt{2}+\sin\alpha\right) y_0^2 v_u^2\,, \tag{29}$$

$$m_\pi^2 - m_\sigma^2 = \pm\frac{1}{\sqrt{3}}\cos\left(\alpha \mp \frac{\pi}{3}\right)\left[\sqrt{2} - \sin\left(\alpha \mp \frac{\pi}{\sqrt{3}}\right)\right] y_0^2 v_u^2\,, \tag{30}$$

$$m_\eta^2 - m_\sigma^2 = \mp\frac{1}{\sqrt{3}}\cos\left(\alpha \pm \frac{\pi}{3}\right)\left[\sqrt{2} - \sin\left(\alpha \pm \frac{\pi}{3}\right)\right] y_0^2 v_u^2\,, \tag{31}$$

where $|\alpha| < \pi/2$. For the case with the normal hierarchy, we should take the upper signs in (29)–(31), so that we obtain

$$m_\eta^2 < m_\sigma^2 < m_\pi^2 \quad \text{for} \quad -\frac{\pi}{6} < \alpha < \frac{\pi}{6}\,. \tag{32}$$

For the case with the inverse hierarchy, since we should take the lower signs in (29)–(31), we obtain

$$m_\pi^2 < m_\eta^2 < m_\sigma^2 \quad \text{for} \quad -\frac{\pi}{2} < \alpha < -\frac{\pi}{6}\,. \tag{33}$$

Next, let us seek the numerical value of α that gives the ratio of the observed values of $\Delta m_{\text{solar}}^2 = (7.9^{+0.6}_{-0.5}) \times 10^{-5}\,\text{eV}^2$ [42, 43] and $\Delta m_{\text{atm}}^2 = (2.74^{+0.44}_{-0.26}) \times 10^{-3}\,\text{eV}^2$ [44, 45],

$$R_{\text{obs}} \equiv \frac{\Delta m_{\text{solar}}^2}{\Delta m_{\text{atm}}^2} = (2.9 \pm 0.5) \times 10^{-2}. \tag{34}$$

The predicted ratio $R(\alpha)$ is given by

$$R(\alpha) \equiv \frac{m_\sigma^4(\alpha) - m_\eta^4(\alpha)}{m_\pi^4(\alpha) - m_\sigma^4(\alpha)}\,, \tag{35}$$

for the normal hierarchy with $m_\eta^2 < m_\sigma^2 \ll m_\pi^2$. From $R(\alpha) = R_{\text{obs}}$, we find

$$\alpha = \left(3.0^{-1.2}_{+1.4}\right)^\circ, \tag{36}$$

where the $\mp$ sign corresponds to the $\pm$ sign of the experimental error in (34). Similarly, we considered the case with the inverse hierarchy, but we find that there is no solution with the inverse hierarchy.

The solution $\alpha = (3.0^{-1.2}_{+1.4})^\circ$ gives

$$\begin{aligned} m_\eta &= -\left(0.076^{+0.006}_{-0.008}\right) y_0 v_u\,, \\ m_\sigma &= (0.38 \pm 0.01) y_0 v_u\,, \\ m_\pi &= (0.923 \mp 0.006) y_0 v_u\,. \end{aligned} \tag{37}$$

The result $m_\eta < 0$ leads to the change of the sign $\sqrt{m_{\nu 1}} \to -\sqrt{m_{\nu 1}}$ in a relation similar to (5):

$$m_{\nu 1} + m_{\nu 2} + m_{\nu 3} = \frac{2}{3}\left(-\sqrt{m_{\nu 1}} + \sqrt{m_{\nu 2}} + \sqrt{m_{\nu 3}}\right)^2. \tag{38}$$

The relation (38) for the neutrino masses has recently been speculated on by Brannen [46] by the algebraic method (however, the algebraic method is highly mathematical, and the physical meaning of this method is somewhat unclear in the "masses and mixings").

The values (37) predict the following neutrino masses:

$$\begin{aligned} m_{\nu 1} &= (3.5 \pm 0.5) \times 10^{-4}\,\text{eV}\,, \\ m_{\nu 2} &= (8.7 \pm 0.2) \times 10^{-3}\,\text{eV}\,, \\ m_{\nu 3} &= \left(5.23^{-0.25}_{+0.40}\right) \times 10^{-2}\,\text{eV}\,, \end{aligned} \tag{39}$$

from the input value $m_{\nu 3} = \sqrt{\Delta m_{\text{atm}}^2}$.

Generally, the masses m_{fi} that satisfy the relation (5) [or (38)] are expressed by a bilinear form,

$$m_{fi} = (z_{fi})^2 m_{f0}\,, \tag{40}$$

where the sector-dependent parameters z_{fi} are normalized by $(z_{f1})^2 + (z_{f2})^2 + (z_{f3})^2 = 1$. Then the parameters z_{fi} can always be expressed in the form

$$\begin{aligned} z_{f1} &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\sin\xi_f\,, \\ z_{f2} &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\sin\left(\xi_f + \frac{2}{3}\pi\right), \\ z_{f3} &= \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{3}}\sin\left(\xi_f + \frac{4}{3}\pi\right), \end{aligned} \tag{41}$$

where we have taken $z_{f1}^2 < z_{f2}^2 < z_{f3}^2$. From the observed charged lepton mass values [15], we obtain the numerical value of ξ_e:

$$\xi_e = \frac{\pi}{4} - \varepsilon = 42.7324^\circ \quad (\varepsilon = 2.2676^\circ)\,. \tag{42}$$

Note that in the limit of $\varepsilon \to 0$ the electron mass becomes zero. We consider the parameter ε to be a fundamental parameter, which governs the charged lepton mass spectrum.

Comparing the expression (27) (with the upper signs) with the expression (41), we find that the parameter α is connected to ξ_ν by the relation

$$\alpha = \frac{\pi}{3} - \xi_\nu \,. \tag{43}$$

Therefore, we obtain

$$\xi_\nu - \xi_e = \left(\frac{\pi}{3} - \alpha\right) - \left(\frac{\pi}{4} - \varepsilon\right) = \frac{\pi}{12} + \varepsilon - \alpha \,. \tag{44}$$

Since the value of α of (36), which is a solution of $R(\alpha) = R_{\rm obs}$, is very close to the value $\varepsilon = 2.27°$, (42), which follows from the observed charged lepton masses, we can regard α as being equal to ε, $\alpha = \varepsilon$. Then we obtain a phenomenological relation:

$$\xi_\nu = \xi_e + \frac{\pi}{12} \,. \tag{45}$$

The relation (45) has also been speculated on by Brannen [46], but the reason for this equation is still controversial. (Of course, in the present model, there is no theoretical reason for the equality $\alpha = \varepsilon$.)

4 Case with $y_0^2 + y_1^2 = y_2^2$

In the previous section, we have assumed the constraint (25) on the Yukawa coupling constants y_0, y_1 and y_2. However, the theoretical basis of this constraint is not clear. In the present section, instead of the constraint (25), we assume another constraint:

$$y_0^2 + y_1^2 = y_2^2 + y_3^2 \,, \tag{46}$$

which is again satisfied by the Yukawa coupling constants (16) in the charged lepton sector. The condition (46) means the requirement of universality of the coupling constants in an extended meaning: the coupling constants of $\bar{\psi}_a \psi_a$ $(a = \pi, \eta, \sigma)$ to the scalars are normalized with equal weights for the scalars ϕ_σ and ϕ_π (ϕ_η).

In the neutrino sector, since we have assumed $y_3 = 0$, we can denote the condition (46) as

$$y_0 = y_2 \cos\beta \,, \quad y_1 = y_2 \sin\beta \,. \tag{47}$$

Then the mass eigenvalues (24) are expressed as follows:

$$\begin{aligned} m_\pi &= \frac{1}{2}[\sin(\beta + \phi_0) \pm 1]|y_2|v_u \,, \\ m_\eta &= \frac{1}{2}[\sin(\beta + \phi_0) \mp 1]|y_2|v_u \,, \\ m_\sigma &= \frac{1}{\sqrt{2}} \cos(\beta + \phi_0) y_2 v_u \,, \end{aligned} \tag{48}$$

where

$$\sin\phi_0 = \sqrt{\frac{2}{3}} \,, \quad \cos\phi_0 = \frac{1}{\sqrt{3}} \quad (\phi_0 = 54.74°) \,; \tag{49}$$

we have again taken the condition (22), i.e.

$$\sin(\beta + \phi_0) > 0 \quad (-\phi_0 < \beta < \pi - \phi_0) \,, \tag{50}$$

and the upper and lower signs in (48) correspond to the cases $y_2 v_\eta > 0$ (normal hierarchy case) and $y_2 v_\eta < 0$ (inverse hierarchy case), respectively.

From the expression (48), we find

$$m_\pi^2 + m_\eta^2 + m_\sigma^2 = y_2^2 v_u^2 \,, \tag{51}$$

$$m_\eta + m_\sigma + m_\pi = \sqrt{\frac{3}{2}} y_2 v_u \cos\beta \,. \tag{52}$$

Therefore, we obtain

$$\frac{\frac{2}{3}(m_\pi + m_\eta + m_\sigma)^2}{m_\pi^2 + m_\eta^2 + m_\sigma^2} = \cos^2\beta = 1 - \sin^2\beta \,. \tag{53}$$

Thus, the parameter β in the present model denotes a deviation from the mass formula (38) [(28)].

Note that if we find a solution $\beta = \beta_1$ that gives $R(\beta) = R_{\rm obs}$ [$R(\beta)$ is given by (35) with $\alpha \to \beta$, and $R_{\rm obs}$ is given by (34)], then the value $\beta_2 = 2\phi_0 - \beta_1$ [ϕ_0 is defined by (49)] is also a solution of $R(\beta) = R_{\rm obs}$. From the expression (48), it is obvious that the solutions β_1 and β_2 give the same values for m_π and m_η, but they give values with mutually opposite signs for m_σ. We list those solutions of $R(\beta) = R_{\rm obs}$ in Table 1, together with the values of m_η, m_σ and m_π.

In Table 1, we also list the predicted values of the neutrino masses $m_{\nu 1} = m_\eta^2/M_N$, $m_{\nu 2} = m_\sigma^2/M_N$ and $m_{\nu 3} = m_\pi^2/M_N$ (M_N is the Majorana mass, $M_N \equiv M_{N_1} = M_{N_2} = M_{N_3}$, of the heavy neutrinos N_i). Here, as input value we have used $m_{\nu 3} = \sqrt{\Delta m^2_{\rm atm}} = 0.0523\,{\rm eV}$ for the case of the normal hierarchy, and $m_{\nu 2} = \sqrt{\Delta m^2_{\rm atm}} = 0.0523\,{\rm eV}$ for the case of the inverse hierarchy. At present, the numerical

Table 1. Solutions of $R(\beta) = R_{\rm obs}$

β	$m_\eta/y_2 v_u$	$m_\sigma/y_2 v_u$	$m_\pi/y_2 v_u$	$m_{\nu 1}$ [eV]	$m_{\nu 2}$ [eV]	$m_{\nu 3}$ [eV]
2.94°	−0.0775	0.3781	0.9225	0.000368	0.00877	0.0523
67.59°	−0.0775	−0.3781	0.9225	0.000368	0.00877	0.0523
− 35.64°	0.6636	0.6682	−0.3364	0.0515	0.0523	0.0132
106.17°	0.6636	−0.6682	−0.3364	0.0515	0.0523	0.0132

values of $m_{\nu i}$ should not be taken rigidly. Therefore, we have omitted the error values in Table 1.

5 Neutrino mixing matrix

As we discussed in Sect. 2, the additional rotation $R(\theta_{\pi\eta})$ from the tribimaximal mixing, (18), depends only on the value v_π^u/v_η^u, and it is independent of the values of y_0, y_1 and y_2. In order to see the effects of the additional rotation $R(\theta_{\pi\eta})$, we change from the basis $(\nu_\pi, \nu_\eta, \nu_\sigma)$ defined by (4) to the basis $(\nu_\eta, \nu_\sigma, \nu_\pi)$ given by

$$\begin{pmatrix}\nu_\eta\\ \nu_\sigma\\ \nu_\pi\end{pmatrix} = U_{\mathrm{TB}}^{\mathrm{T}}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}, \tag{54}$$

where U_{TB} is the tribimaximal mixing matrix defined by (1). If $v_\pi/\nu_\eta \neq 0$, i.e. $R(\theta_{\pi\eta}) \neq \mathbf{1}$, the neutrino mixing matrix U_ν is given by

$$U_\nu = U_{\mathrm{TB}}\begin{pmatrix} c_{\pi\eta} & 0 & s_{\pi\eta}\\ 0 & 1 & 0\\ -s_{\pi\eta} & 0 & c_{\pi\eta}\end{pmatrix}$$
$$= \begin{pmatrix} -\frac{2}{\sqrt{6}}c_{\pi\eta} & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}}s_{\pi\eta}\\ \frac{1}{\sqrt{6}}c_{\pi\eta}+\frac{1}{\sqrt{2}}s_{\pi\eta} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}s_{\pi\eta}-\frac{1}{\sqrt{2}}c_{\pi\eta}\\ \frac{1}{\sqrt{6}}c_{\pi\eta}-\frac{1}{\sqrt{2}}s_{\pi\eta} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}s_{\pi\eta}+\frac{1}{\sqrt{2}}c_{\pi\eta}\end{pmatrix}, \tag{55}$$

where $s_{\pi\eta} = \sin\theta_{\pi\eta}$ and $c_{\pi\eta} = \cos\theta_{\pi\eta}$, i.e.

$$\tan^2\theta_{\mathrm{solar}} = \frac{1}{2c_{\pi\eta}^2}, \tag{56}$$

$$\sin^2 2\theta_{\mathrm{atm}} = \left(1-\frac{4}{3}s_{\pi\eta}^2\right)^2, \tag{57}$$

$$(U_\nu)_{13}^2 = \frac{2}{3}s_{\pi\eta}^2. \tag{58}$$

For convenience, we define the following z_i-parameters:

$$\langle\phi_i^u\rangle = z_i^u v_u, \quad \langle\phi_i^d\rangle = z_i^d v_d, \tag{59}$$

with the normalization $\sum_i (z_i^u)^2 = \sum_i (z_i^d)^2 = 1$. Here, note that in (34) we have already defined the z_{fi}-parameters similarly to the present z_i^u- and z_i^d-parameters. In the charged lepton sector, since $\sqrt{m_{ei}} \propto v_i^d$, the relation $z_{ei} = z_i^d$ holds. However, in the neutrino sector, since m_{L}^ν is not diagonal, the values $z_{\nu i}$ are not identical with z_i^u.

For the z_i^d-parameters, from the relation (5) we obtain

$$\frac{z_1^d}{\sqrt{m_e}} = \frac{z_2^d}{\sqrt{m_\mu}} = \frac{z_3^d}{\sqrt{m_\tau}} = \frac{1}{\sqrt{m_e+m_\mu+m_\tau}}, \tag{60}$$

i.e.

$$z_1^d = 0.016473, \quad z_2^d = 0.23678, \quad z_3^d = 0.97140. \tag{61}$$

If we assume $z_i^u = z_i^d$, we obtain $z_\pi^u = 0.51939$, $z_\eta^u = 0.47982$ and $z_\sigma^u = 1/\sqrt{2}$ from the definition (4). Then the rotation angle $\theta_{\pi\eta} = -(1/2)\tan^{-1}(v_\pi^u/v_\eta^u) = -23.63°$ is too large to explain the observed neutrino mixings [see (56)–(58)], so that the case $z_i^u = z_i^d$ is ruled out.

By the way, it is well known that the so-called $2 \leftrightarrow 3$ symmetry [47–51] is promising for the neutrino mass matrix description. Therefore, the simplest assumption is to require the $2 \leftrightarrow 3$ symmetry for the VEV values $\langle\phi_i^u\rangle$, i.e. $v_2^u = v_3^u$, which leads to

$$v_\pi^u = 0. \tag{62}$$

The case gives $\theta_{\pi\eta} = 0$ from (19), so that the neutrino mixing is exactly given by the tribimaximal mixing (1). Note that if we required the $2 \leftrightarrow 3$ symmetry for the fields $\ell_{\mathrm{L}} = (\nu_{\mathrm{L}}, e_{\mathrm{L}})$, the symmetry would affect the charged lepton sector too. Here, we have required the $2 \leftrightarrow 3$ symmetry only for $\langle\phi_i^u\rangle$, not for $\langle\phi_i^d\rangle$, so that the symmetry does not affect the charged lepton mass matrix.

Of course, the $2 \leftrightarrow 3$ symmetry is a phenomenological requirement, and the constraint may be broken. From the observed constraint [52], $(U_\nu)^2 < 0.03$, we obtain the constraint $|\theta_{\pi\eta}| < 12.2°$, i.e.

$$\left|\frac{v_\pi}{v_\eta}\right| < 0.46. \tag{63}$$

6 Concluding remarks

In conclusion, based on the universal seesaw mass matrix model (6) with three scalars ϕ_i, and by assuming S_3 flavor symmetry for the Yukawa interactions, we have investigated the neutrino masses and mixings. For the VEV values of ϕ_i^f $(f = u, d)$, stimulated by the Higgs potential model [39, 40] for ϕ_i, we have assumed the constraint

$$\langle\phi_\pi^f\rangle^2 + \langle\phi_\eta^f\rangle^2 = \langle\phi_\sigma^f\rangle^2, \tag{64}$$

where $(\phi_\pi, \phi_\eta, \phi_\sigma)$ are defined by (4). However, since we have four independent Yukawa coupling constants, y_0, y_1, y_2 and y_3, which are defined by (15), the model does not have predictability. Therefore, in the present paper, as suggested by the observed neutrino mixing (the tribimaximal mixing), we have investigated only the simple case with $y_3 = 0$, where only the ν_π–ν_η mixing is caused. (Since the observed neutrino mixing is not the exact tribimaximal mixing, the condition $y_3 = 0$ is only an approximate requirement; it is taken for convenience.) In the case of $y_3 = 0$ together with the assumption (64), our conclusion is as follows: the mass eigenvalues depend only on the values of the coupling constants y_0, y_1 and y_2, while the ν_π–ν_η mixing angle, $\theta_{\pi\eta}$, depends only on the value of $\langle\phi_\pi^u\rangle/\langle\phi_\eta^u\rangle$. Therefore, we can discuss the topic of the neutrino mass spectrum independently from that of the deviation from tribimaximal mixing.

For the neutrino mass spectrum, from an economical point of view as regards the number of parameters, we have investigated two typical cases: with the constraints $y_0^2 = y_1^2 + y_2^2$ and $y_0^2 + y_1^2 = y_2^2$. The former case leads to the case which satisfies Brannen's relation (38) for the neutrino masses. Although the relation (38) is very interesting, the theoretical basis of the constraint $y_0^2 = y_1^2 + y_2^2$ is not clear. On the other hand, the latter case is likely from the point of view of universality of the coupling constants. The latter case does not satisfy the relation (38). Only for a small value of the parameter β, the deviation from the relation (38) may become negligibly small. For example, for the solution $\beta = 2.94°$ given in Table 1, the deviation from the relation (38) is very small, $\sin^2\beta = 0.003$, as seen in (53), so that the relation (38) is approximately satisfied.

The neutrino mixing matrix U_ν can lead to tribimaximal mixing (1) in the limit of $v_2^u = v_3^u$, as given in (55). If we require $2 \leftrightarrow 3$ symmetry for the v_i^u (not for the v_i^d), we can obtain the tribimaximal mixing (1) without affecting the charged lepton mass spectrum. Of course, the $2 \leftrightarrow 3$ symmetry is a phenomenological requirement, and the constraint may be broken. From the observed constraint [52], $(U_\nu)^2 < 0.03$, we obtain the constraint $|v_\pi/v_\eta| < 0.46$.

The numerical predictions of $m_{\nu i}$ shown in (39) and in Table 1 were obtained by adjusting the parameter α (β) for the observed ratio $\Delta m^2_{\rm solar}/\Delta m^2_{\rm atm}$. Finally, we would like to make a remark on the speculation on the neutrino masses of assuming the simple Yukawa interaction form [53] (and without using the observed value of $\Delta m^2_{\rm solar}/\Delta m^2_{\rm atm}$)

$$H_\nu = y_\nu\left(\frac{\bar\ell_\pi N_\pi + \bar\ell_\eta N_\eta + \bar\ell_\sigma N_\sigma}{\sqrt{3}}\phi^u_\sigma + \frac{\bar\ell_\pi N_\eta + \bar\ell_\eta N_\pi}{\sqrt{2}}\phi^u_\pi + \frac{\bar\ell_\pi N_\pi - \bar\ell_\eta N_\eta}{\sqrt{2}}\phi^u_\eta\right). \quad (65)$$

Here, in the charged lepton sector, we have assumed universality of the coupling constants on the basis (e_1, e_2, e_3), while, in the neutrino sector, we have assumed universality of those on the S_3 irreducible basis $(\nu_\pi, \nu_\eta, \nu_\sigma)$. Then the neutrino masses are predicted to be

$$\begin{aligned} m_{\nu1} &= \left(\frac{1}{\sqrt{6}} - \frac{1}{2}\right)^2 m_0^\nu\,, \\ m_{\nu2} &= \frac{1}{6} m_0^\nu\,, \\ m_{\nu3} &= \left(\frac{1}{\sqrt{6}} + \frac{1}{2}\right)^2 m_0^\nu\,, \end{aligned} \quad (66)$$

without an adjustable parameter. This case predicts

$$R = \frac{\Delta m^2_{21}}{\Delta m^2_{32}} = \frac{4\sqrt{6}-9}{4\sqrt{6}+9} = 0.0425\,. \quad (67)$$

The value (67) is somewhat large compared to the observed value (34), but, at present, this case is not ruled out within three sigma. Again, regarding $m_{\nu3}$ as $m_{\nu3} = \sqrt{\Delta m^2_{\rm atm}}$, we predict the explicit neutrino mass values to be as follows:

$$\begin{aligned} m_{\nu1} &= \left(5.3^{+0.4}_{-0.3}\right) \times 10^{-4}\ \text{eV}\,, \\ m_{\nu2} &= \left(1.05^{+0.07}_{-0.05}\right) \times 10^{-2}\ \text{eV}\,, \\ m_{\nu3} &= \left(5.22^{+0.35}_{-0.25}\right) \times 10^{-2}\ \text{eV}\,. \end{aligned} \quad (68)$$

The case (65) is also interesting because of the simplicity of its structure. The predictions (67) and (68) should be taken as results in the ideal limit.

In conclusion, the present model (a lepton mass matrix model with a bilinear form) based on S_3 symmetry has given many interesting features for the mass spectra and mixings. However, the model still includes some adjustable parameters. Further investigation, based on another symmetry that gives stronger constraints on the parameters than those in the case of the S_3 symmetry, will be desired.

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Appendix

In order to see whether the mass hierarchy is $m_\eta^2 < m_\sigma^2 < m_\eta^2$ or $m_\pi^2 < m_\eta^2 < m_\sigma^2$, we estimate the differences among those masses as follows:

$$m_\pi^2 - m_\eta^2 = \pm\frac{1}{\sqrt{3}}|y_2|\left(\sqrt{2}y_0 + y_1\right)v_u^2\,, \quad (A.1)$$

$$m_\pi^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}\left(\sqrt{3}y_1 \pm |y_2|\right)\left(2\sqrt{2}y_0 - y_1 \pm \sqrt{3}|y_2|\right)v_u^2\,, \quad (A.2)$$

$$m_\eta^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}\left(\sqrt{3}y_1 \mp |y_2|\right)\left(2\sqrt{2}y_0 - y_1 \mp \sqrt{3}|y_2|\right)v_u^2\,. \quad (A.3)$$

Since we have defined the factor $(\sqrt{2}y_0 + y_1)$ as positive in (22), (A.1) means that, for the case of the normal hierarchy, with $m_\pi^2 > m_\eta^2$, we must take the upper signs in (A.2) and (A.3), i.e.

$$m_\pi^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}\left(\sqrt{3}y_1 + |y_2|\right)\left(2\sqrt{2}y_0 - y_1 + \sqrt{3}|y_2|\right)v_u^2 > 0\,, \quad (A.4)$$

$$m_\eta^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}\left(\sqrt{3}y_1 - |y_2|\right)\left(2\sqrt{2}y_0 - y_1 - \sqrt{3}|y_2|\right)v_u^2 < 0\,, \quad (A.5)$$

and, for the case of the inverse hierarchy, with $m_\pi^2 < m_\eta^2$, we must take the lower signs in (A.2) and (A.3), i.e.

$$m_\pi^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}\left(\sqrt{3}y_1 - |y_2|\right)\left(2\sqrt{2}y_0 - y_1 - \sqrt{3}|y_2|\right)v_u^2 < 0\,, \quad (A.6)$$

$$m_\eta^2 - m_\sigma^2 = \frac{1}{4\sqrt{3}}\left(\sqrt{3}y_1 + |y_2|\right)\left(2\sqrt{2}y_0 - y_1 + \sqrt{3}|y_2|\right)v_u^2 < 0\,. \quad (A.7)$$

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